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Pascal's Triangle and Sierpinski's Triangle: An Incredible Link

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At first glance, Pascal's triangle and Sierpinski's triangle appear to share nothing in common other than their triangular shape. Consider the difference in time period when the two formations were discovered. Pascal's triangle (an algebraic mnemonic) was used by him in the mid-1600's, while Sierpinski's triangle (a fractal) became widely researched during the boom of the computer age in the 1970's. Centuries separate their origin, but there is a simple mathematical link between the two triangles.

Blaise Pascal was born in 1623 and died in 1662, a short but brilliant life. Pascal was a child prodigy who began attending meetings with senior French mathematicians at the age of fourteen (Dunham, 1990). Pascal, at the age of sixteen, submitted scholar-like material to Rene Descartes (Richardson & Richardson, 1973). Descartes was so impressed with Pascal's writings that he refused to believe such sophisticated and thoughtful work came from such a young person (Dunham, 1990).

Waclaw Sierpinski, a Polish mathematician, is best known for his work with fractals (Gulick, 1992). Sierpinski has been given credit for the discoveries of two interesting and substantial formations, the Sierpinski triangle and the Sierpinski carpet (Gulick). The Sierpinski carpet is a simple formation that becomes complicated as it advances through its developmental stages.

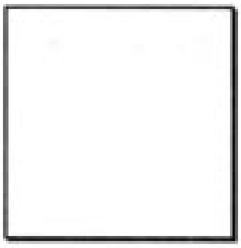
Fractals and Construction by Iteration

The Sierpinski carpet and the Sierpinski triangle are fractals formed by a process called iteration. When we decide we are going to perform a function's operations over and over (for specific values of x), we output the iterates of that function (Gulick , 1992). For example, if we wanted the 5th iterate (where x = 0 is the first iterate, and x = 1 is the second iterate, and so on) of the function $f(x) = 2^x$:

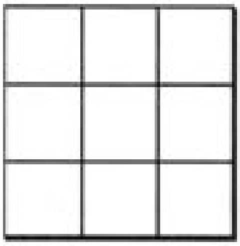
$$f(x) = 2^x \Rightarrow f(4) = 2^4 = 16$$

The iteration process is slightly more involved when forming a fractal such as the Sierpinski carpet. Let us look at the step-by-step process of the formation of the Sierpinski carpet.

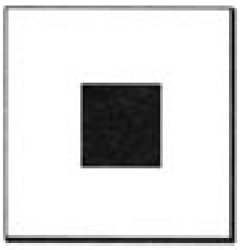
1. The Sierpinski carpet starts off with a square $[n = 0]$.



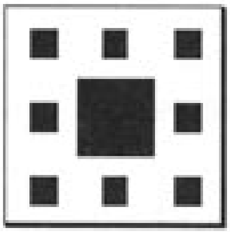
2. We then divide each side of the figure into three equal parts, connecting the points in the following fashion.



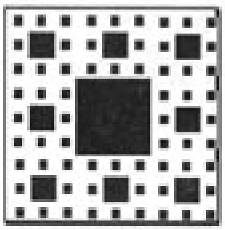
3. This process divides the square into 9 smaller squares; now we must remove the middle square. This completes one iteration $[n = 1]$.



4. Now we must start the entire process over, performing the same process to each of the remaining 8 squares individually. This process quickly becomes complicated and time consuming $[n=2]$.



5. After one more iteration, the pattern is quite evident. However, it is approaching the limit of printing capabilities.



Sierpinski's carpet and triangle are considered fractals for two reasons. One, the process is infinite, and two, there is a self similarity within the figure (i.e., if you zoom in on the first tiny square at the top left portion of the carpet, the carpet would appear once again just as in the previous illustration). The iteration process continues to infinity, so the picture is limited only by the graphical capability of the source producing the carpet.

Pascal's Triangle: Simple Yet Unlimited in Scope

In a glance at Pascal's triangle, one sees the obvious characteristics of the numerical structure: the numbers start at 1 and get large rapidly. Let us observe the structure of Pascal's triangle:

> *For the things of this world cannot be made known without a knowledge of mathematics.*
>
> Roger Bacon
>
> *Opus Majus* part 4 *Distinctia Prima cap 1*, 1267.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 89 364 901 2002 3003 3432 3003 2002 901 364 89 14 1

1 15 103 453 1265 2903 5005 6435 6435 5005 2903 1265 453 103 15 1

1 16 118 556 1718 4168 7908 11440 12870 11440 7908 4168 1718 556 118 16 1

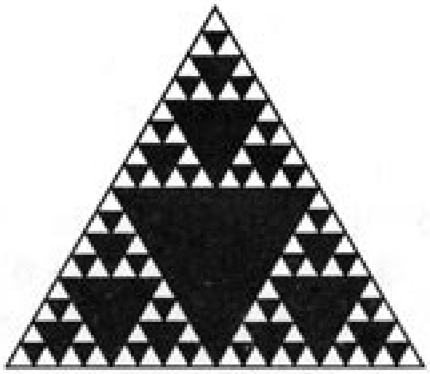
And so on...

After a closer look, one can see that each entry in the body of the triangle is obtained by adding the numbers in the row above to the left and the right (Dunham, 1990). For example, in the 13th row, 792 is the result of adding 330 and 462, the numbers above 792 on the left and right. This explains why there are 1's all the way down each side. On the left side, each new "1" is formed by adding the number above and to the right (1), and to the left (nothing).

The Sierpinski Triangle: If You Have Seen One Portion, You Have Seen It All

The Sierpinski triangle is formed in a way similar to Sierpinski's carpet with respect to an infinite deleting of objects within the structure. However, the rules of the construction are slightly different from the ones that govern the formation of the carpet.

Since the Sierpinski triangle is a fractal, it stems from a series of iterations that dictate its characteristics and appearance (Gulick, 1992). We start with a solid, equilateral triangle. (The triangle need not be equilateral, however its appearance is sometimes more attractive in this form.) The first step in the "shape deleting" process involves bisecting each of the triangle's three sides. Then we must remove the central triangle that is formed by these three points. The deleting of the central triangle marks the end of the first iteration. Now we must perform the same iteration process to all three of the resulting triangles. After a few iterations, the pattern is quite obvious and intricate.



The Sierpinski carpet and the Sierpinski triangle theoretically are less than two-dimensional (Gulick, 1992). The formations are on the page for us to observe, in black and white (two-dimensionally), yet they are not two-dimensional in theory (shapes like circles, squares, and triangles which lie in a plane are considered two-dimensional). The black portions of the Sierpinski carpet and triangle represent emptiness or nothingness. These portions are not a different colored part of the fractal. They represent absence within the formation. Let us see how mathematics allowed us to make this discovery.

If we let "*N*" equal the number of triangles present within the main triangle, and if we let "*L*" equal the length of the sides of the remaining triangles, the Fractal Dimension (*D*) of the Sierpinski triangle will be as follows:

$$D = \lim_{n \to \infty} \frac{\log N(n)}{\log[1 / L(n)]}$$

Since $N(0) = 1$, $N(1) = 3$, $N(2) = 9$, $N(3) = 27$, ... , then $N(n) = 3^n$.

Also, since $L(1) = \frac{1}{2}$, and $L(2) = \frac{1}{4}$, ... , then $L(n) = \left(\frac{1}{2}\right)^n$.

Therefore, $D = \lim_{n \to \infty} \frac{\log 3^n}{\log\left[\frac{1}{(\frac{1}{2})^n}\right]} = \frac{\log 3}{\log 2} \approx 1.58496$.

In a similar way, one can show that the Sierpinski carpet has a fractal dimension $\frac{\ln 8}{\ln 3} \approx 1.89279$. So, these fractal formations have a theoretical dimension of less than two! This characteristic is due to the infinite deleting that occurs during iteration.

The dimension of these fractals is just one more aspect of this mathematical puzzle. There is a simple manipulation that will intimately link all of these interesting mathematical concepts together.

If one considers Pascal's triangle and Sierpinski's triangle, the initial assumption might be that these two triangular formations share their shape as the sole similarity. However, this is not so. One might ask, "How could two formations from totally different origins, times, and backgrounds be intimately linked?" Through a very simple manipulation, we are going to see just how structurally similar these triangles are.

The transformation is something like a game. There are two rules to the construction. Considering Pascal's Triangle:

1. When a given number is odd, replace it with a "1."
2. When a given number is even, replace it with a "0."

After completing the first two rows, no correspondence is apparent. However, after completion of several lines, similarity in structure to Sierpinski's triangle is the intriguing result!

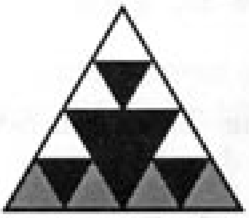
1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1
1 1 0 0 0 0 0 0 1 1
1 0 1 0 0 0 0 0 1 0 1
1 1 1 1 0 0 0 0 1 1 1 1
1 0 0 0 1 0 0 0 1 0 0 0 1
1 1 0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

And so on...

It seems that Pascal's triangle takes on the structure of Sierpinski's triangle when it is reformed using the simple manipulation technique presented above. What does this imply?

The Mathematical Link

There is a mathematical link between these two formations. The expression 2^n has a direct and consistent connection to both Pascal's and Sierpinski's triangle (Jurgens, 1992). For example, the $n = 2$ iteration of Pascal's triangle (the 3rd row) has a sum of 4, which equals 2^2. In the $n = 2$ iteration of Sierpinski's triangle, after "2 sets" of triangles have been deleted, there are 4 triangles remaining in the bottom row (smallest triangles remaining) of the triangle, which equals 2^2.



n = 2

1

1 1

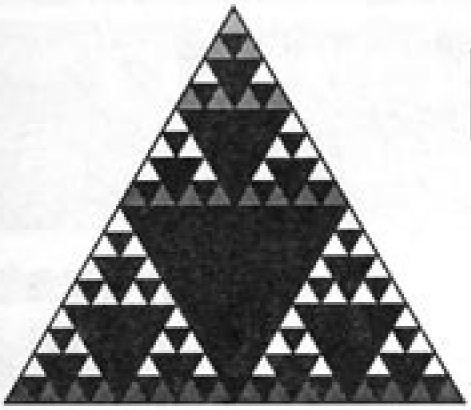
1 2 1

n = 2

This pattern continues through many iterations. For example, the $n = 5$ iteration of Pascal's triangle (the 6th row) has a sum of 32, which equals 2^5. In the $n = 5$ iteration of Sierpinski's triangle, after "5 sets" of triangles have been deleted, there are 32 triangles remaining in the bottom row (smallest triangles remaining) of the triangle, which equals 2^5.

Further inspection of the Sierpinski triangle reveals an additional link to Pascal's triangle. If you consider the rows of remaining triangles that are fully "connected" (All of the triangles in the row share a corner vertex with another triangle, on each side, other than the end triangles.), you will find representations for the sums of the rows for Pascal's triangle from $n = 0$ to whatever iteration of Sierpinski's triangle that you observe. Here is an illustration of this link using Sierpinski's triangle at $n = 4$.

> *I advise my students to listen carefully the moment they decide to take no mathematics course. They might be able to hear the sound of closing doors.*
>
> *James Caballero*
>
> "Everybody a mathematician?", *CAIP Quarterly, 2*, Fall, 1989.



Pascal's triangle n=?	Sum of Row
0	$2^n = 2^0 = 1$
1	$2^n = 2^1 = 2$
2	$2^n = 2^2 = 4$
3	$2^n = 2^3 = 8$
4	$2^n = 2^4 = 16$

It is true that both of the triangles display the "halving" characteristic that logically relates them to the expression 2^n. However, it is remarkable that these two formations, so different in origin and appearance (other than their triangular shape), are so intimately linked.

This incredible link can be shared with mathematics students in classes as basic as elementary algebra. These students can learn about Pascal's triangle to expand a binomial raised to a power. After demonstrating the usefulness of Pascal's triangle, one could convey the concept of Sierpinski's triangle and the mathematical expression that categorizes the triangular concepts as "intimately linked."

References

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Richardson, M. & Richardson, L.F. (1973). *Fundamentals of mathematics*. New York: Macmillan.

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

Galileo Galilei (1564-1642)

Operell Saggiatore, p. 171.